Shape reconstruction using multifrequency topological derivatives

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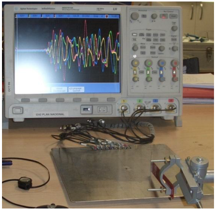
Outline

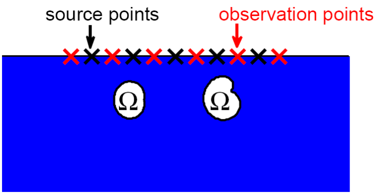
Description of the problem

Medium $\mathcal{R}$ with obstacles Ω: How many? how big? where?



Some applications

- Medicine (tumors, fracture)
- Geophysics (oil, gas)
- Materials (damage, cracks)

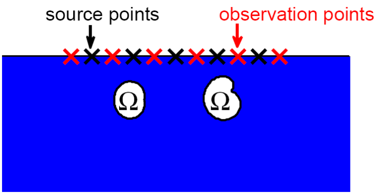


Scattering problem

Produce an image of the medium generating an incident wave and recording the received wave

Forward (direct) problem

- The shape, size, location and physical properties of the objects are known
- Generate an acoustic wave at some source points
- Compute the response of the system at the detectors "×"
- A well–posed problem: it has a unique solution that depends continuously on the data



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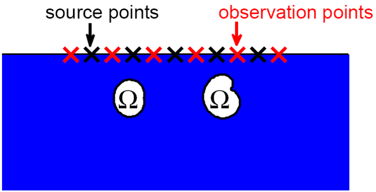

Scattering problem

Produce an image of the medium generating an incident wave and recording the received wave

Inverse problem

- The total wave $U_{meas}(\mathbf{x}, t)$ is measured at the observation points
- Find the regions Ω s.t.

  $U = U_{meas}$ at the obs. points, U= sol. forward problem
- An ill–posed problem: it may not have a solution and if it has one, it may not depend continuously on the data

Scattering problem

We assume that

- We generate acoustic waves

$$\rho U_{tt} = k\Delta U + f$$

- Incident waves are time–harmonic

$$U_{inc}(\mathbf{x}, t) = \mathrm{Re}[e^{-i\omega t} u_{inc}(\mathbf{x})], \quad \omega = \text{frequency},$$

- u_{inc} is generated at $\mathbf{x}_0$
- The solution to the direct problem is time–harmonic

$$U(\mathbf{x}, t) = \mathrm{Re}[e^{-i\omega t} u(\mathbf{x})]$$

Forward problem

Ω is a penetrable known obstacle. The incident field generates a scattered wave u_{sc} in $\mathbb{R}^2 \setminus \Omega$ and a transmitted wave u_{tr} in Ω. The total field

$$u = u_{inc} + u_{sc} \ \text{ in } \mathbb{R}^2 \setminus \Omega \quad \text{and} \quad u = u_{tr} \text{ in } \Omega$$

solves

$$\begin{cases} k_e \Delta u + \rho_e \omega^2 u = -\delta_{\mathbf{x}_0} & \text{in } \mathbb{R}^2 \setminus \Omega \\ k_i \Delta u + \rho_i \omega^2 u = 0 \quad \text{in } \Omega \\ u^- = u^+, \quad k_e \partial_n u^- = k_i \partial_n u^+ \quad \text{on } \partial\Omega \\ \lim_{r\to\infty} r^{1/2}(\partial_r (u - u_{inc}) - i\omega\sqrt{k_e/\rho_e}(u - u_{inc})) = 0 \end{cases}$$

where $\omega,\ k_e,\ k_i,\ \rho_e,\ \rho_i > 0$ are known

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1. Inverse scattering problems
2. Topological derivative methods
   - TD for shape reconstruction
   - Numerical examples for one–frequency waves
   - Multifrequency topological derivatives
3. Conclusions
   - Conclusions and current work

Constrained optimization

Original problem

Find Ω such that

$$u = u_{meas} \quad \text{at the observation points } \mathbf{x}_1, \ldots, \mathbf{x}_{N_{obs}}$$

A weaker formulation

Find Ω minimizing

$$J(\Omega) = \frac{1}{2} \sum_{j=1}^{N_{obs}} |u(\mathbf{x}_j) - u_{meas}(\mathbf{x}_j)|^2$$

for u solving the forward problem with objects Ω

- The domain Ω is the variable
- The Helmholtz transmission problem is the constraint

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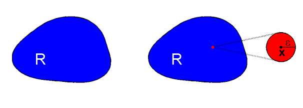
Definition of Topological Derivative (Sokowloski–Zochowski '99)

The TD of a shape functional $J(\mathcal{R})$ at a point $\mathbf{x} \in \mathcal{R}$ is

$$D_T(\mathbf{x}, \mathcal{R}) = \lim_{\varepsilon \to 0} \frac{J(\mathcal{R} \setminus B_\varepsilon(\mathbf{x})) - J(\mathcal{R})}{\text{Vol}(B_\varepsilon(\mathbf{x}))}$$



- It is a scalar function of $\mathbf{x}$
- It measures sensitivity to removing balls around $\mathbf{x}$
- $D_T(\mathbf{x}, \mathcal{R}) << 0 \implies$ high probability of finding an object

Equivalently, for $\mathbf{x} \in \mathcal{R}$ and $h(\varepsilon) = \text{Vol}(B_\varepsilon(\mathbf{x}))$

$$J(\mathcal{R} \setminus B_\varepsilon(\mathbf{x})) = J(\mathcal{R}) + h(\varepsilon) D_T(\mathbf{x}, \mathcal{R}) + o(h(\varepsilon)) \quad \text{as } \varepsilon \to 0$$

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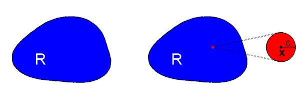

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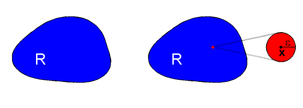

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Theorem (Carpio & Rapún '08)

For any $\mathbf{x} \in \mathbb{R}^2$ the topological derivative of

$$J(\Omega) = \frac{1}{2} \sum_{j=1}^{N_{obs}} |u(\mathbf{x}_j) - u_{meas}(\mathbf{x}_j)|^2$$

is

$$D_T(\mathbf{x}) = \mathrm{Re}\left[\omega^2(\rho_i - \rho_e)\, u(\mathbf{x}) p(\mathbf{x}) + \frac{2k_e(k_e - k_i)}{k_e + k_i} \nabla u(\mathbf{x}) \cdot \nabla p(\mathbf{x})\right]$$

where u and p solve forward and adjoint problems with $\Omega = \emptyset$

Forward problem with $\Omega = \emptyset$:

$$\begin{cases} k_e \Delta u + \omega^2 \rho_e u = -\delta_{\mathbf{x_0}} & \text{in } \mathbb{R}^2 \\ \lim_{r\to\infty} r^{1/2}(\partial_r(u - u_{inc}) - i\omega\sqrt{k_e/\rho_e}(u - u_{inc})) = 0 \end{cases}$$

Therefore, $u(\mathbf{x}) = u_{inc}(\mathbf{x}) = \dfrac{i}{4k_e} H_0^{(1)}(\omega\sqrt{k_e/\rho_e}|\mathbf{x} - \mathbf{x}_0|)$

Adjoint problem with $\Omega = \emptyset$:

$$\begin{cases} k_e \Delta p + w^2 \rho_e p = \sum_{j=1}^{N_{obs}} (\overline{u_{meas} - u})\delta_{\mathbf{x}_j} & \text{in } \mathbb{R}^2 \\ \lim_{r\to\infty} r^{1/2}(\partial_r p - i\omega\sqrt{k_e/\rho_e} p) = 0 \end{cases}$$

Therefore,

$$p(\mathbf{x}) = \sum_{j=1}^{N_{obs}} \frac{i}{4k_e} H_0^{(1)}(\omega\sqrt{k_e/\rho_e}|\mathbf{x} - \mathbf{x}_j|)(\overline{u(\mathbf{x}_j) - u_{meas}(\mathbf{x}_j)})$$

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Remarks:

- The true obstacles enter in the TD through the measured data at the adjoint field
- Very cheap computation:

$$D_T(\mathbf{x}) = \mathrm{Re}\left[\omega^2(\rho_i - \rho_e)\, u(\mathbf{x})p(\mathbf{x}) + \frac{2k_e(k_e - k_i)}{k_e + k_i}\nabla u(\mathbf{x}) \cdot \nabla p(\mathbf{x})\right]$$

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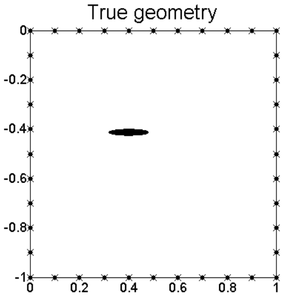
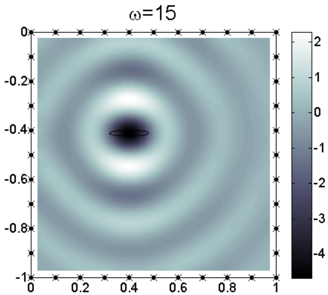
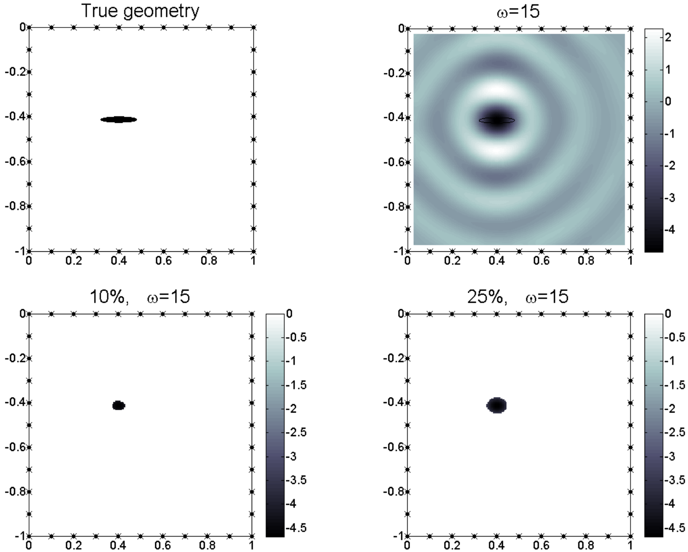
Example 1





- "$\times$"= emisor/receptor points
- Level of noise=1%
- $k_e = \rho_e = 1$, $k_i = 1/2$, $\rho_i = 1/50$ ($\lambda_e/\lambda_i = 5$, $\lambda := \omega\sqrt{k/\rho}$).

Example 1: emisors/receptors at 4 sides



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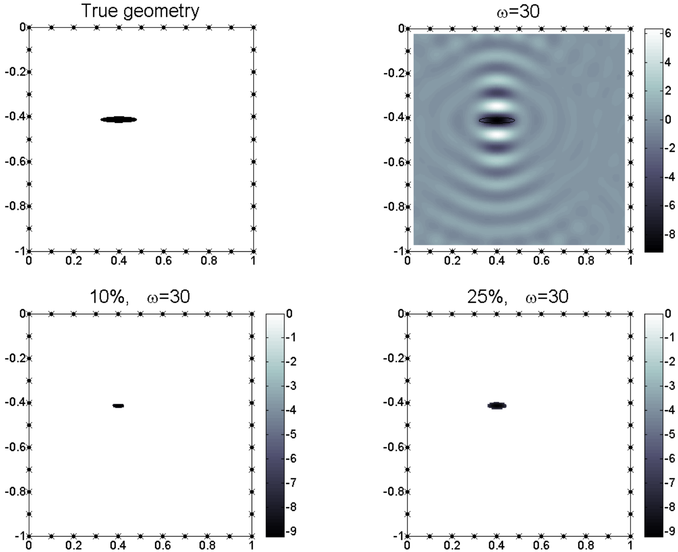

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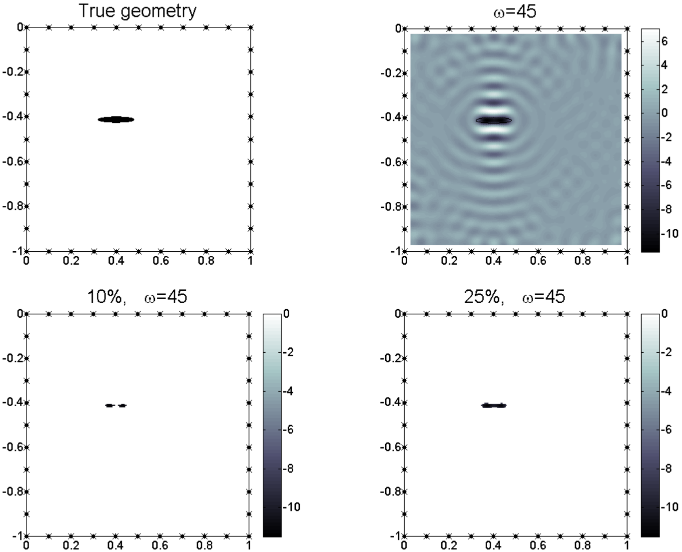

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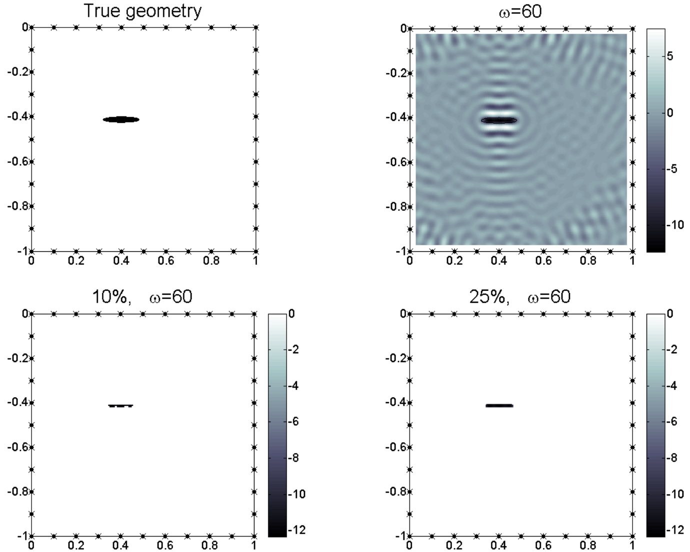

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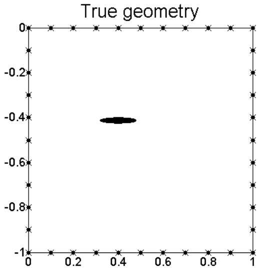
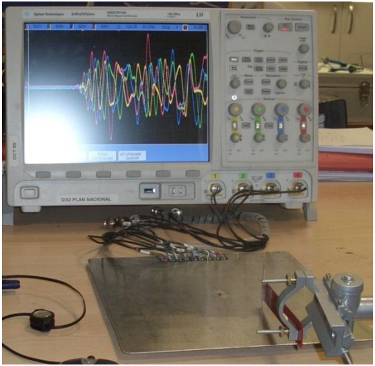


Remarks

- Only one defect
- Not always the higher frequencies provide the best results.
- An ideal situation: emisors/receptors around the object

Example 2: emisors/receptors at 1 side



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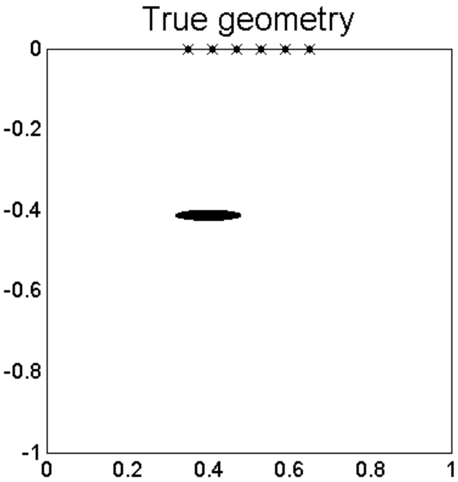

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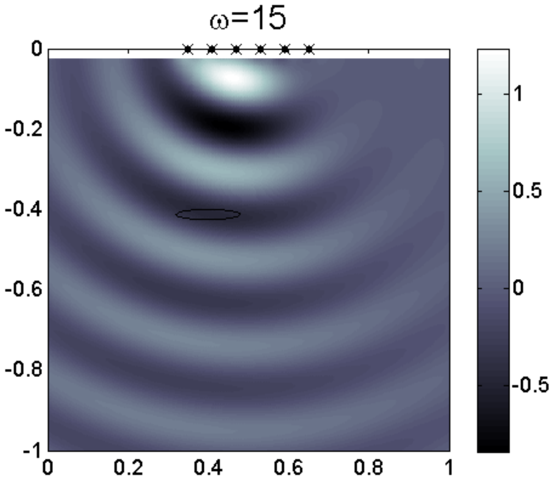

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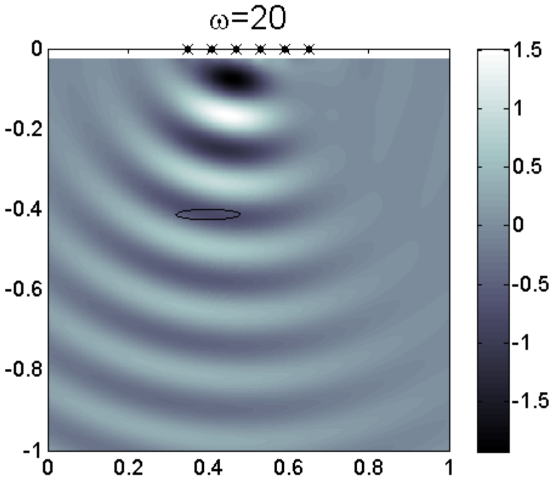

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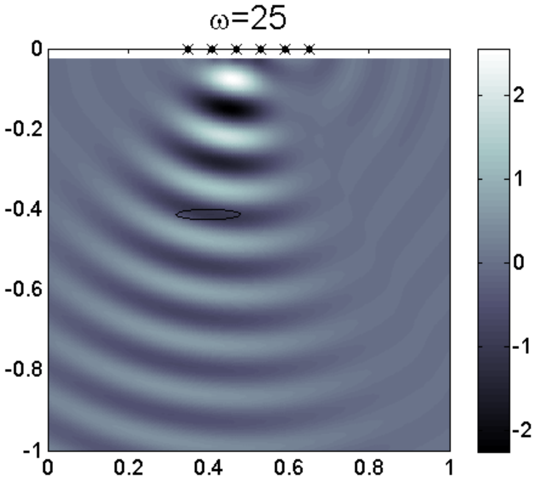

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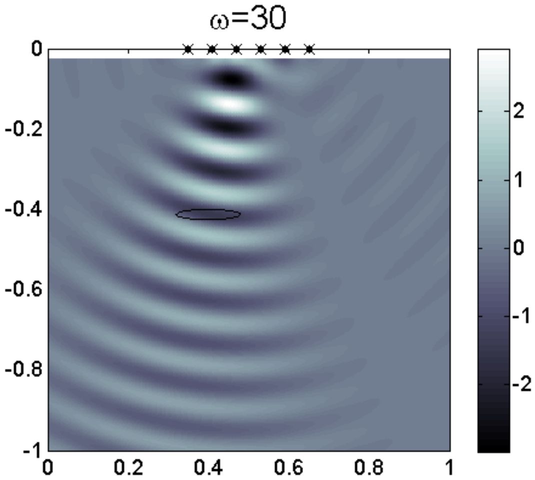

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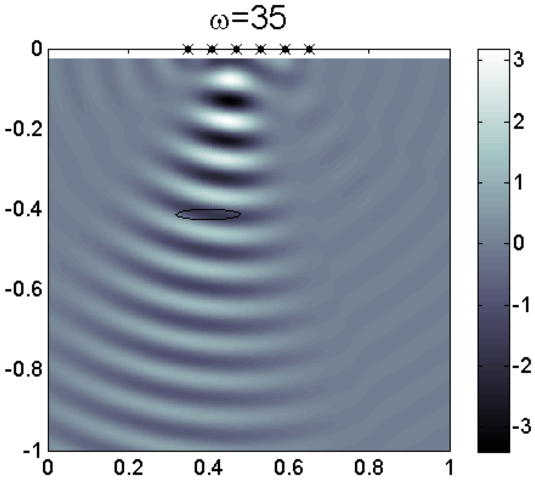

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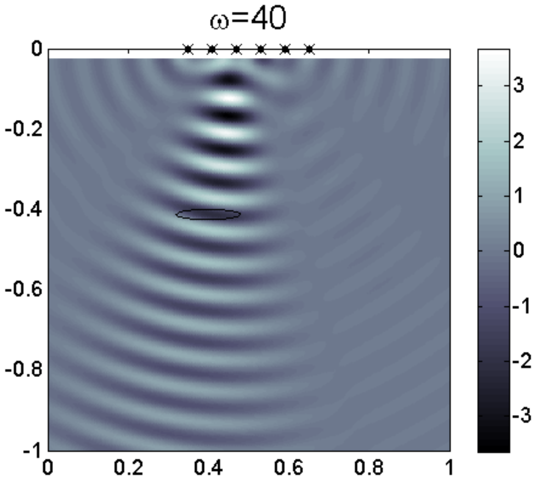

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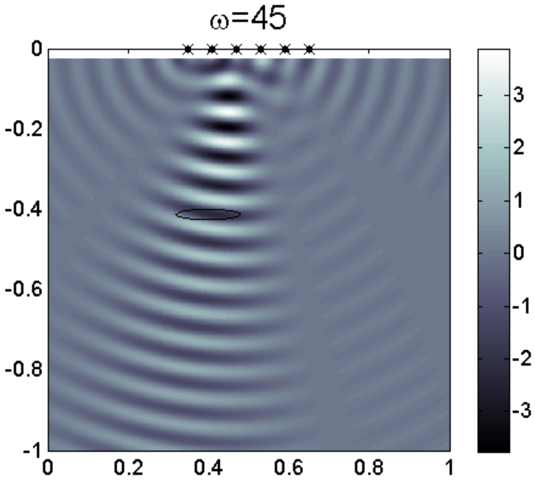

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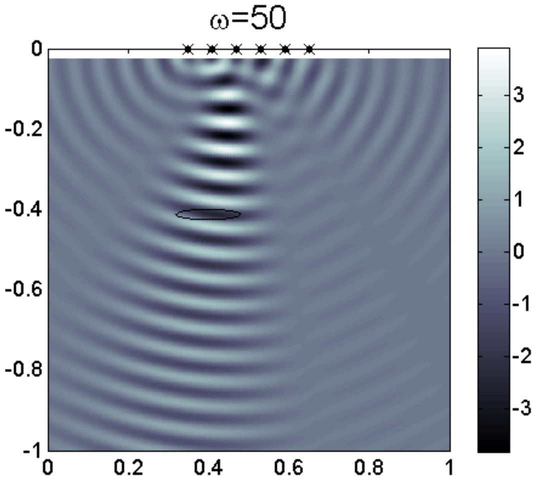

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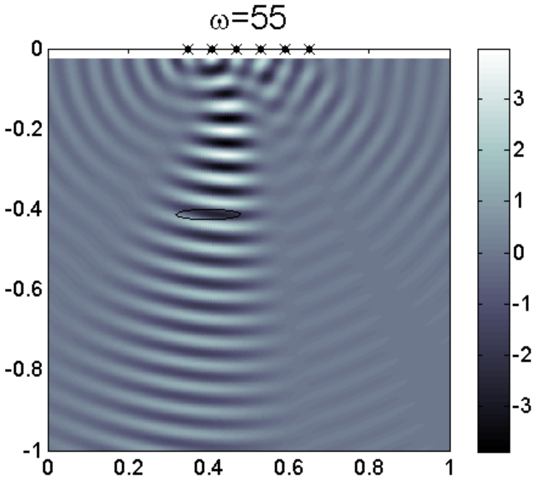

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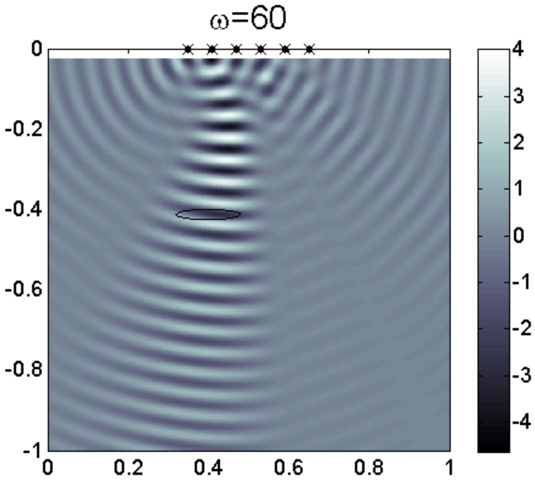

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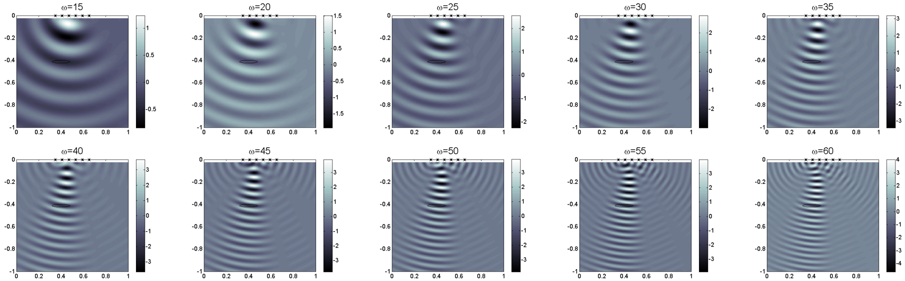
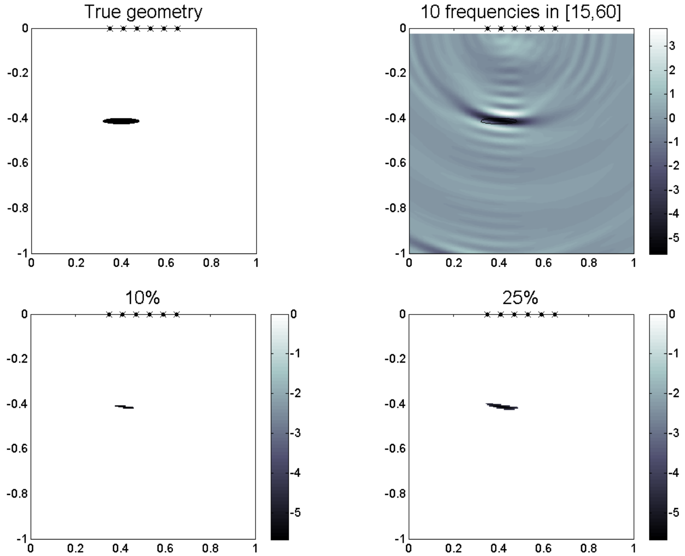
Multifrequency topological derivative



Idea

Can we combine one–frequency results to recover the object?

Weighted topological derivative



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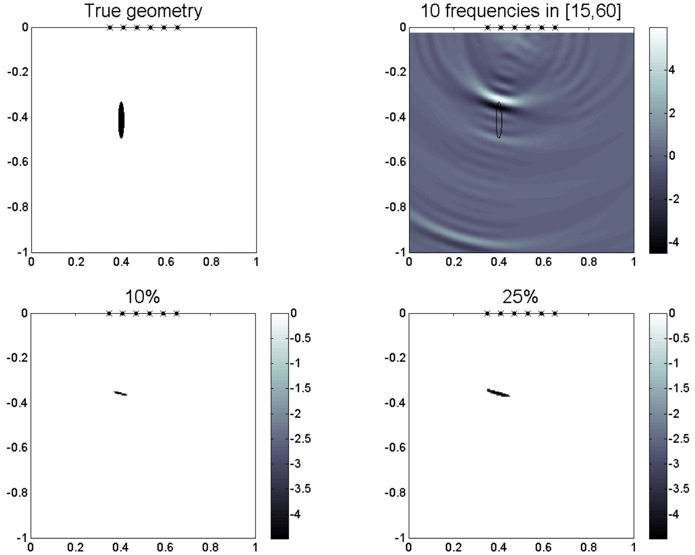

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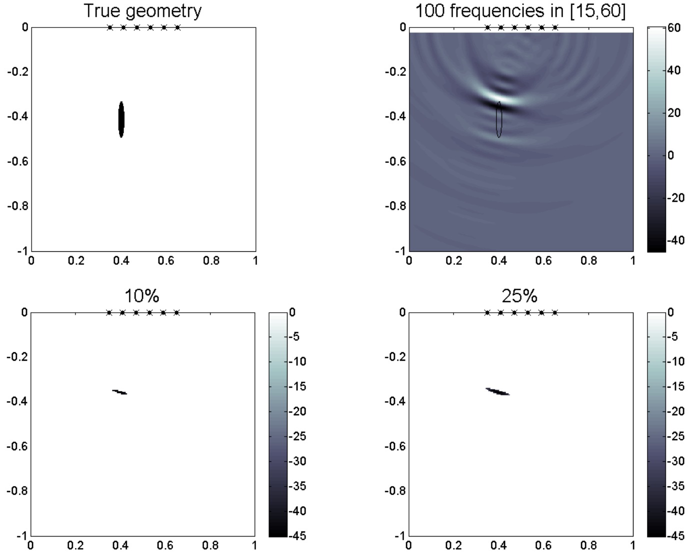

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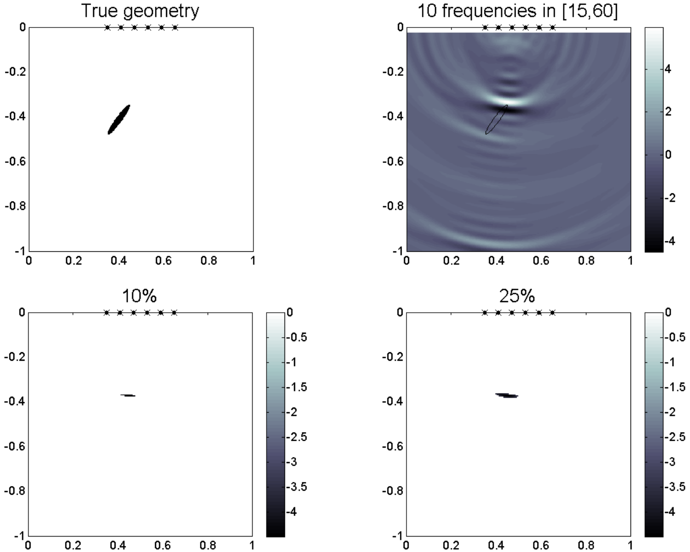

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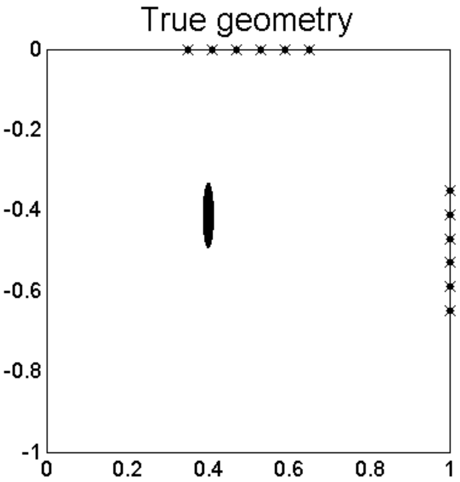

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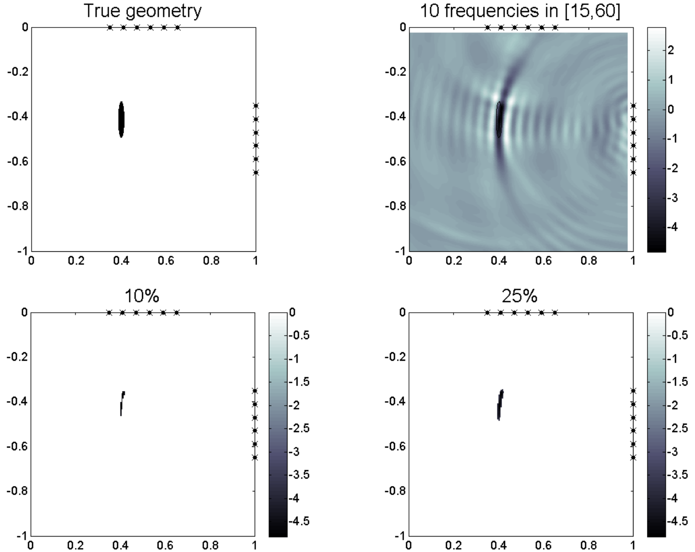

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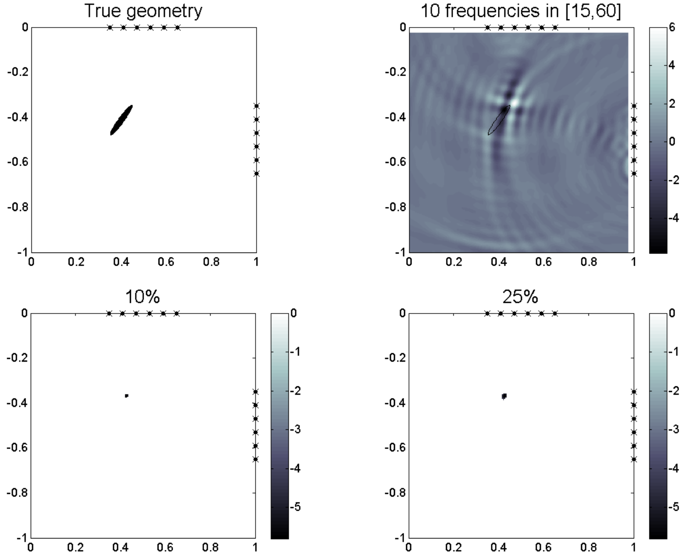

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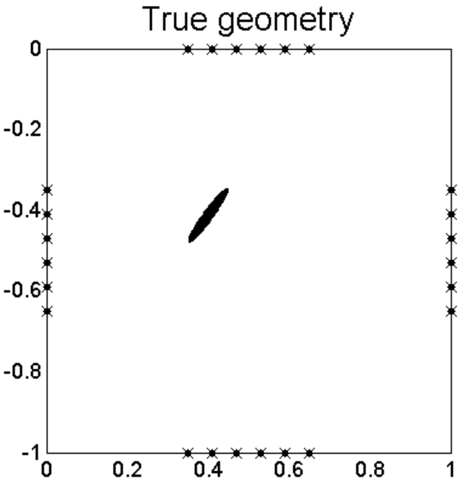

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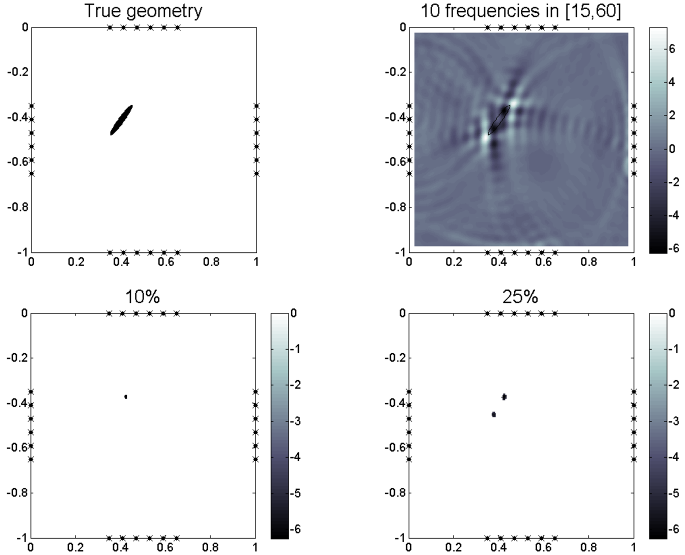

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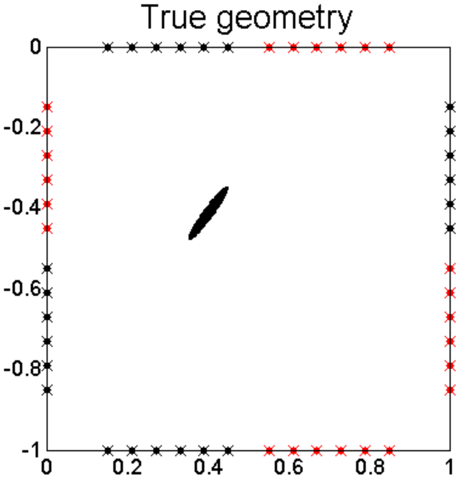

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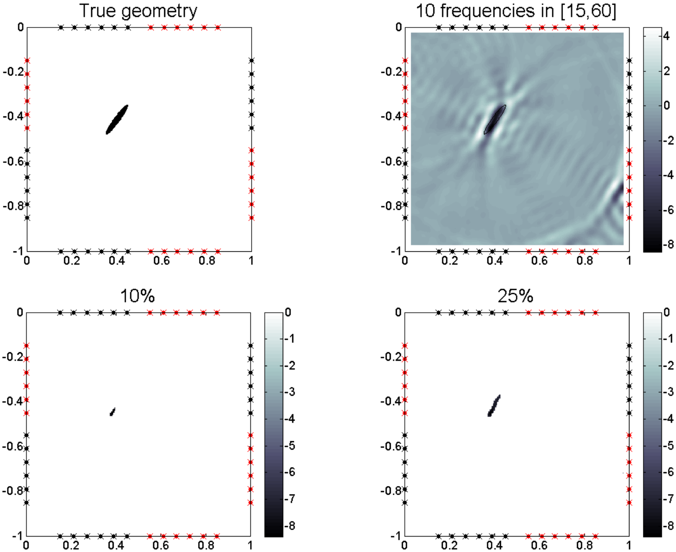
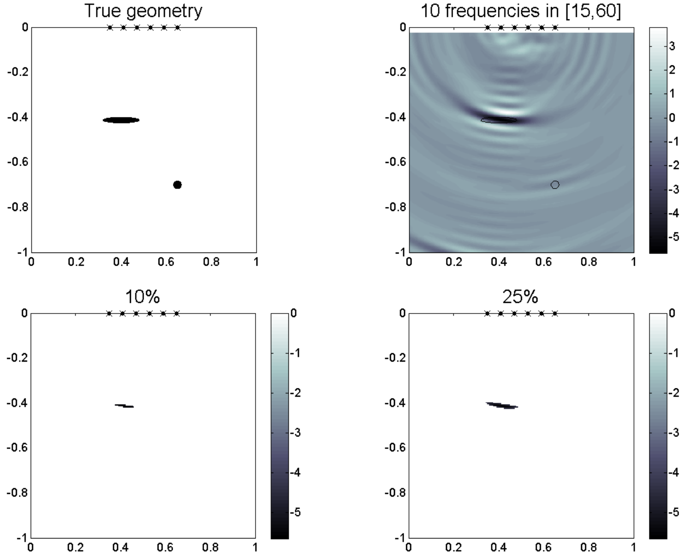


Multiple objects



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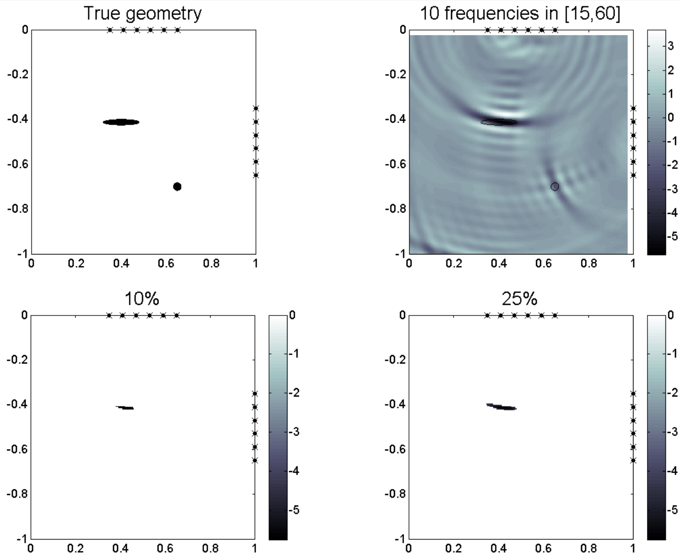
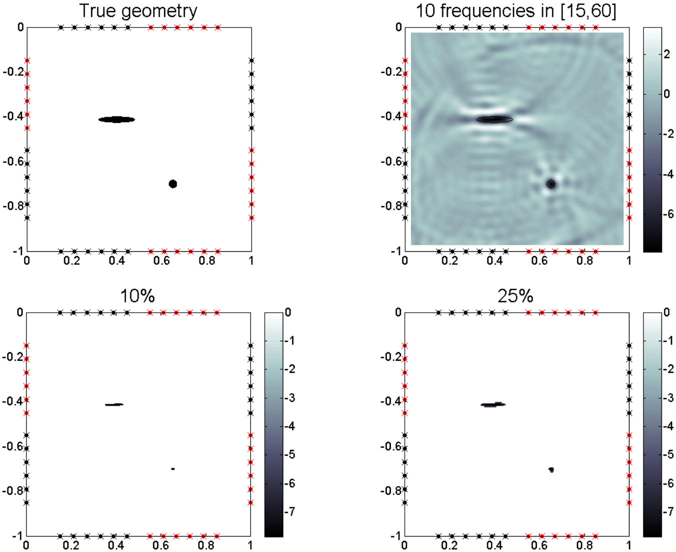
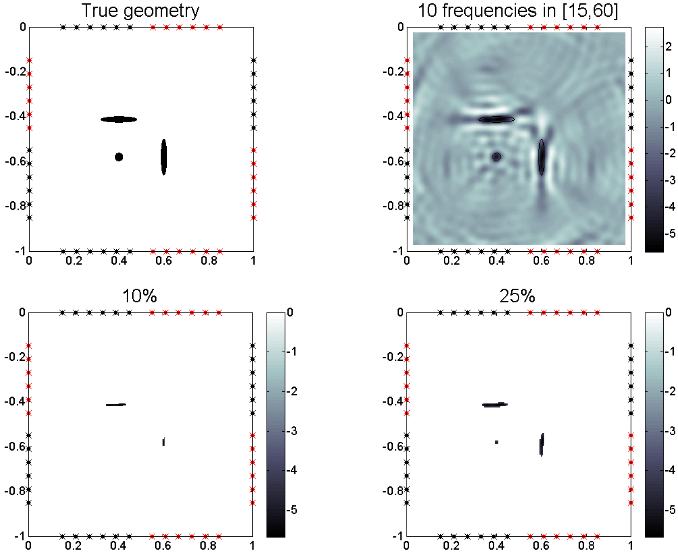


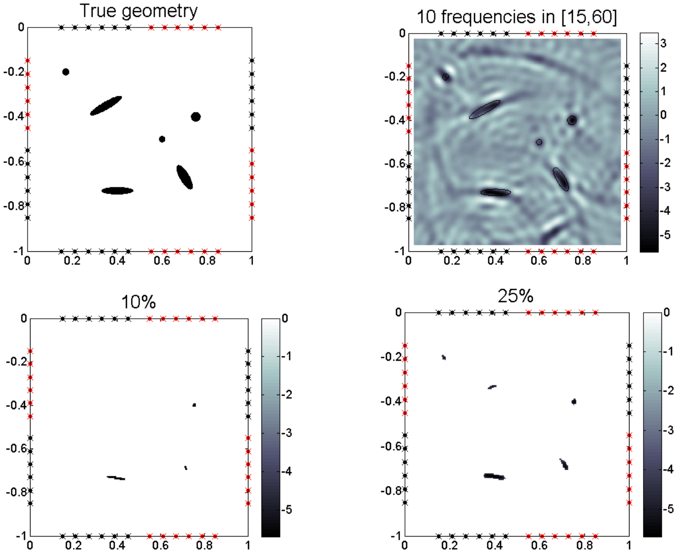
Multiple objects



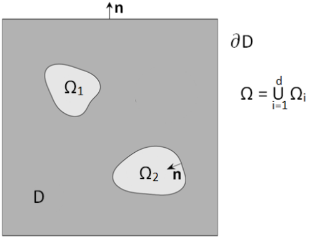
Multiple objects



Multiple objects



Neumann problem in a finite domain



$$\begin{cases} k\Delta u + \rho\omega^2 u = -\delta_{\mathbf{x}_0} & \text{in } D \setminus \Omega \\ \partial_n u = 0 & \text{on } \partial\Omega \\ \partial_n u = 0 & \text{on } \partial D \end{cases}$$

Neumann problem in a finite domain

Theorem (Funes, Perales, Rapún & Vega '14)

For any $\mathbf{x} \in D$ the topological derivative of

$$J(\Omega) = \frac{1}{2} \sum_{j=1}^{N_{obs}} |u(\mathbf{x}_j) - u_{meas}(\mathbf{x}_j)|^2$$

is

$$D_T(\mathbf{x}) = \mathrm{Re}\left[-\omega^2 \rho\, u(\mathbf{x}) p(\mathbf{x}) + 2k\, \nabla u(\mathbf{x}) \cdot \nabla p(\mathbf{x})\right]$$

where u and p solve forward and adjoint problems with $\Omega = \emptyset$

Forward problem with $\Omega = \emptyset$:

$$\begin{cases} k\Delta u + \omega^2 \rho u = -\delta_{\mathbf{x_0}} & \text{in } D \\ \partial_{\mathbf{n}} u = 0 & \text{on } \partial D \end{cases}$$

Remark: u is known analytically

Adjoint problem with $\Omega = \emptyset$:

$$\begin{cases} k\Delta p + w^2 \rho p = \sum_{j=1}^{N_{obs}} (\overline{u_{meas} - u})\delta_{\mathbf{x}_j} & \text{in } D \\ \partial_{\mathbf{n}} p = 0 & \text{in } \partial D \end{cases}$$

Remark: p is also known analytically

Forward problem with $\Omega = \emptyset$:

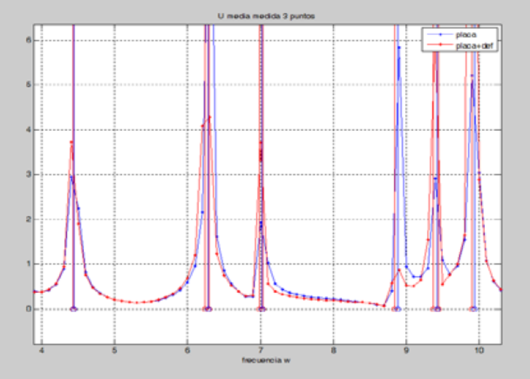
$$\begin{cases} k\Delta u + \omega^2 \rho u = -\delta_{\mathbf{x_0}} & \text{in } D \\ \partial_{\mathbf{n}} u = 0 & \text{on } \partial D \end{cases}$$

Remark: u is known analytically

Adjoint problem with $\Omega = \emptyset$:

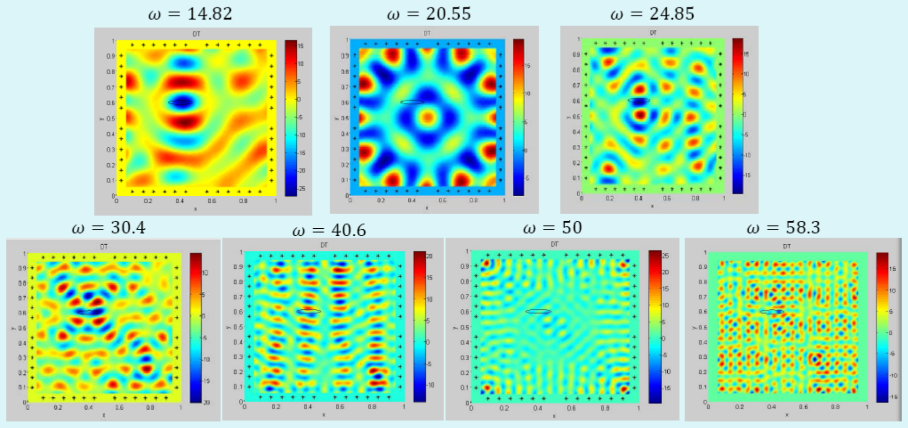
$$\begin{cases} k\Delta p + w^2 \rho p = \sum_{j=1}^{N_{obs}} (\overline{u_{meas} - u}) \delta_{\mathbf{x}_j} & \text{in } D \\ \partial_{\mathbf{n}} p = 0 & \text{in } \partial D \end{cases}$$

Remark: p is also known analytically

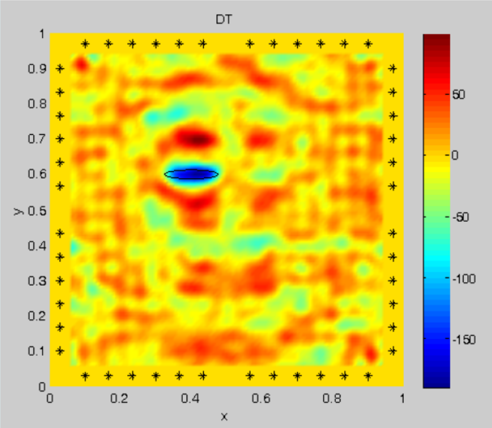


We have to be careful with eigenfrequencies!

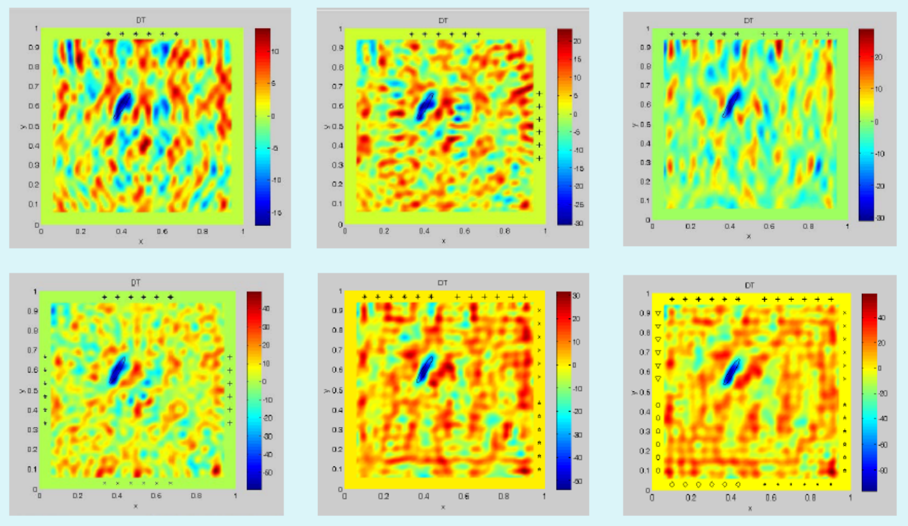
One frequency topological derivatives



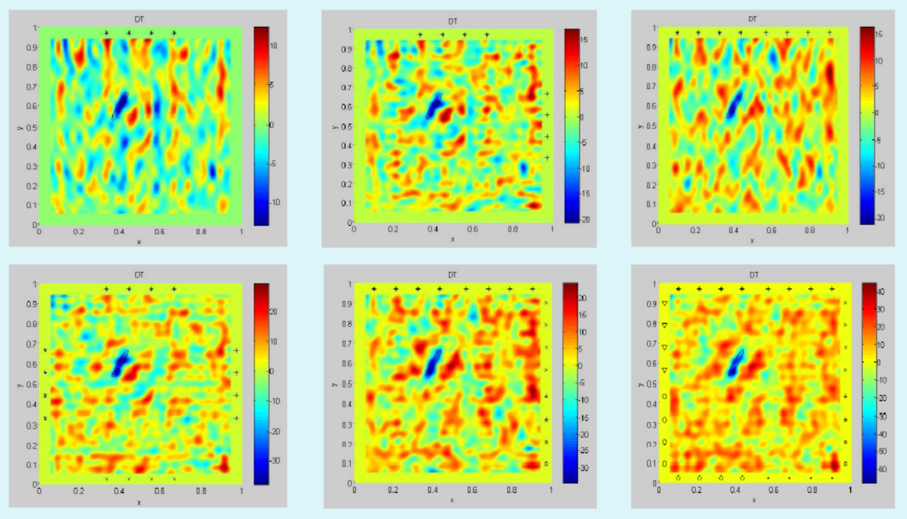
Multifrequency topological derivative



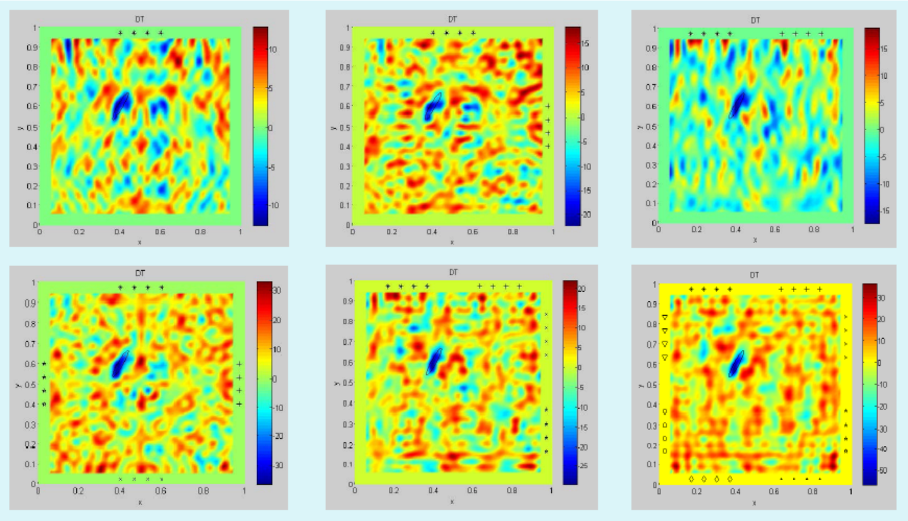
Configuration restrictions: 6 points



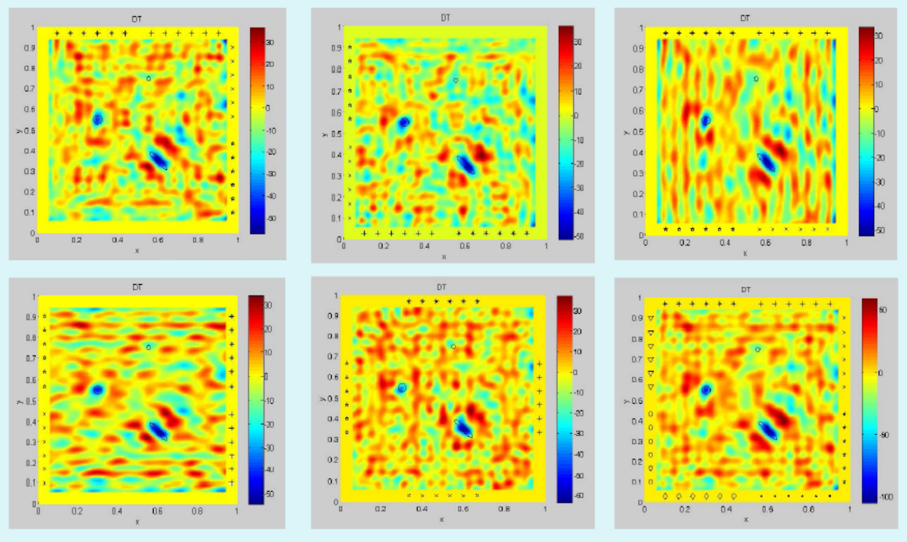
Configuration restrictions: 4 points



Configuration restrictions: 4 close points



Configuration restrictions: multiple defects



Outline

Conclusions and current work

- The topological derivative is a powerful tool to solve inverse problems dealing with shape reconstruction in different areas: acoustics, photothermal problems, elasticity, tomography,...
- The TD gives a good approximation of the number, size and location of the objects buried in a medium
- Multifrequency topological derivative performs much better than one–frequency TD when a few observation/emision points located in a small area are used
- Iterative procedures improve the reconstructions and detect small objects missed in the first trial (in progress)
- Different functionals (different weights) can be considered (in progress)